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Thermal instability in rotating galactic coronae

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ABSTRACT
The thermal stability of rotating, stratified, unmagnetized atmospheres is studied by means of linear-perturbation analysis, finding stability, overstability or instability, depending on the properties of the gas distribution, but also on the nature of the perturbations. In the relevant case of distributions with outward-increasing specific entropy and angular momentum, axisymmetric perturbations grow exponentially, unless their wavelength is short enough that they are damped by thermal conduction; non-axisymmetric perturbations typically undergo overstable oscillations in the limit of zero conductivity, but are effectively stabilized by thermal conduction, provided rotation is differential. To the extent that the studied models are representative of the poorly constrained hot atmospheres of disc galaxies, these results imply that blob-like, cool overdensities are unlikely to grow in galactic coronae, suggesting an external origin for the high-velocity clouds of the Milky Way.

Key words: hydrodynamics, instabilities, ISM: kinematics and dynamics, ISM: clouds, galaxies: formation

1 INTRODUCTION

Galaxy clusters and massive elliptical galaxies are embedded in hot atmospheres of virial-temperature gas revealed by X-ray observations. Lower-mass galaxies are also believed to have hot gaseous coronae, difficult to detect because the gas is very rarefied. If the coronal gas were thermally unstable (Field 1965) it could fragment into cold gas clouds with substantial implications for the dynamics of cooling flows (Mathews & Bregman 1978; Cowie, Fabian, & Nulsen 1980; Nulsen 1986), but also for galaxy formation and for the origin of the H I high-velocity clouds of the Milky Way (Maller & Bullock 2004; Peek, Putman, & Sommer-Larsen 2008; Kaufmann et al. 2009). In fact, linear-perturbation analysis has shown that the X-ray emitting hot atmospheres of massive elliptical galaxies and galaxy clusters are likely stabilized against thermal instability by a combination of buoyancy and thermal conduction (Malagoli, Rosner & Bodo 1987, hereafter MRB; Balbus & Soker 1989; Tribble 1989). The coronae of disc galaxies are expected to have lower gas temperatures and densities, but their detailed properties are poorly known, because—so far—they eluded detection in X-rays. Nevertheless, their physical parameters are not totally unconstrained: for some nearby massive disc galaxies upper limits to the total X-ray luminosity of the hot halos are available (Rasmussen et al. 2009), while in the special case of the Milky Way different pieces of information can be used to constrain the physical properties of the corona (e.g. Spitzer 1956; Fukugita & Peebles 2006). Studying model coronae consistent with these constraints, Binney, Nipoti & Fraternali (2009, hereafter BNF) showed that also the coronae of disc galaxies are likely thermally stabilized by buoyancy and thermal conduction, at least if rotation of the gas can be neglected. There are no significant constraints on the angular momentum of galactic coronae, but one might expect that they are characterized by at least slow rotation. It is well known that there is an important interplay between rotation and convection (e.g. Tassoul 1978; Balbus 2000, 2001), so rotation might influence the stabilizing effect of buoyancy and then the overall problem of the thermal stability of galactic coronae.

This paper addresses the question of the thermal stability of rotating galactic coronae. For this purpose one needs to study the stability against non-axisymmetric perturbations of a differentially rotating, stratified gas in the presence of thermal conduction and radiative cooling. This is a special case of the general problem of the stability of a rotating stratified gas, which has been widely studied in the astrophysical literature, with specific applications to rotating stars (e.g. Cowling 1951; Goldreich & Schubert 1967; Lebovitz 1967; Fricke 1968; Lynden-Bell & Ostriker 1967; Tassoul 1978; Knobloch & Spruit 1982; Lifschitz & Lebovitz 1993; Menou, Balbus, & Spruit 2004) and accretion discs (e.g. Papaloizou & Pringle 1984; Balbus & Hawley 1991, 1992; Ryu & Goodman 1992; Lin, Papaloizou, & Kley 1993; Brandenburg & Dintrans 2006). The stability-analysis techniques

developed in these previous studies can be applied to the problem addressed in the present paper. In particular, we will perform a linear analysis using Eulerian perturbations, considering both axisymmetric and non-axisymmetric disturbances.

It is reasonable to expect that galactic coronae are magnetized, though there are almost no constraints on the properties of their magnetic fields. For simplicity, in the present work we neglect magnetic fields, but we must bear in mind that they might have some influence on the thermal stability properties of the galactic hot gas, as suggested by previous studies of thermal instability in non-rotating cooling flows (Loewenstein 1990; Balbus 1991, see also BNF for a discussion).

The paper is organized as follows. In Section 2 the problem is set up by recalling the relevant hydrodynamic equations. The thermal-stability analysis is then described for axisymmetric perturbations (Section 3) and for non-axisymmetric perturbations (Section 4). The implications for galactic coronae are discussed in Section 5; Section 6 concludes. A list of commonly used symbols is given for reference in Table 1.

2 THERMAL INSTABILITY IN A ROTATING STRATIFIED GAS: GOVERNING EQUATIONS

A rotating, stratified, unmagnetized atmosphere in the presence of cooling and thermal conduction is governed by the following equations for mass, momentum and energy conservation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0, \tag{1}$$

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\frac{\nabla p}{\rho}-\nabla\Phi, \tag{2}$$

$$\frac{p}{\gamma-1}\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right]\ln(p\rho^{-\gamma})=\nabla\cdot\left(\kappa T^{5/2}\nabla T\right)-\left(\frac{\rho}{\mu m_{\rm p}}\right)^2\Lambda(T). \tag{3}$$

Here ρ, p, T and $\mathbf{v}$ are, respectively, the gas density, pressure, temperature and velocity, Φ is the galactic gravitational potential, $\gamma=5/3$ is the ratio of principal specific heats, Λ is the cooling function, μ is the mean gas particle mass in units of the proton mass $m_{\rm p}$ and κ is the thermal conductivity (Spitzer 1962). As we neglect magnetic fields, we treat the thermal conductivity as a scalar. In this approximation we can at most assume that the actual value of κ is suppressed by tangled magnetic fields below Spitzer's benchmark value (e.g. Binney & Cowie 1981), but of course we cannot account for the effects related to the anisotropic heat transport expected in a magnetized medium (e.g. Balbus 2000; Quataert 2008).

In cylindrical coordinates, assuming axisymmetric gravitational potential, the above hydrodynamic equations write:

$$\frac{\partial\rho}{\partial t}+\frac{1}{R}\frac{\partial R\rho v_R}{\partial R}+\frac{\partial\rho v_z}{\partial z}+\frac{1}{R}\frac{\partial\rho v_\phi}{\partial\phi}=0, \tag{4}$$

$$\frac{\partial v_R}{\partial t}+v_R\frac{\partial v_R}{\partial R}+v_z\frac{\partial v_R}{\partial z}+\frac{v_\phi}{R}\frac{\partial v_R}{\partial\phi}=-\frac{1}{\rho}\frac{\partial p}{\partial R}-\frac{\partial\Phi}{\partial R}+\frac{v_\phi^2}{R}, \tag{5}$$

$$\frac{\partial v_z}{\partial t}+v_R\frac{\partial v_z}{\partial R}+v_z\frac{\partial v_z}{\partial z}+\frac{v_\phi}{R}\frac{\partial v_z}{\partial\phi}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{\partial\Phi}{\partial z}, \tag{6}$$

$$\frac{\partial v_\phi}{\partial t}+v_R\frac{\partial v_\phi}{\partial R}+v_z\frac{\partial v_\phi}{\partial z}+\frac{v_\phi}{R}\frac{\partial v_\phi}{\partial\phi}=-\frac{1}{\rho R}\frac{\partial p}{\partial\phi}-\frac{v_R v_\phi}{R}, \tag{7}$$

$$\frac{p}{\gamma-1}\left[\frac{\partial}{\partial t}+v_R\frac{\partial}{\partial R}+v_z\frac{\partial}{\partial z}+\frac{v_\phi}{R}\frac{\partial}{\partial\phi}\right]\ln(p\rho^{-\gamma})=$$
$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\kappa T^{5/2}\frac{\partial T}{\partial R}\right)+\frac{\partial}{\partial z}\left(\kappa T^{5/2}\frac{\partial T}{\partial z}\right)+\frac{1}{R}\frac{\partial}{\partial\phi}\left(\frac{\kappa T^{5/2}}{R}\frac{\partial T}{\partial\phi}\right)-\left(\frac{\rho}{\mu m_{\rm p}}\right)^2\Lambda(T). \tag{8}$$

The unperturbed atmosphere is assumed to be axisymmetric and close to hydrostatic and thermal equilibrium, in the sense that the system is approximately in a steady state over the time scales of interest even in the presence of radiative cooling and thermal conduction. Thus, the unperturbed fluid can be described by the time-independent axisymmetric pressure p_0, density ρ_0, temperature T_0 and velocity field $\mathbf{v}_0=(v_{0R},v_{0z},v_{0\phi})$, which satisfy equations (4-8) with vanishing partial derivatives with respect to t. The velocity components v_{0R} and v_{0z} are allowed to be non-null, because an inflow of gas can occur if cooling is not perfectly balanced by thermal conduction. Such a steady-state configuration is expected to be a reasonable approximation for the corona sufficiently far from the disc, where we are interested in studying the thermal instability. The model is not intended to apply near the disc and close to the galactic centre, where the interactions of the coronal gas with the star-forming disc and the central super-massive black hole are important.

In the most general case the atmosphere is assumed to be differentially rotating, with angular velocity $\Omega\equiv v_{0\phi}/R$ depending on both R and z, but in the analysis it is convenient to distinguish cases with $\partial\Omega/\partial z=0$ from cases with $\partial\Omega/\partial z\neq0$. The Poincaré-Wavre theorem (e.g. Tassoul 1978) states that the surfaces of constant pressure and constant density coincide if and only if $\partial\Omega/\partial z=0$: as a consequence, distributions with $\partial\Omega/\partial z=0$ are said *barotropic* (pressure is a function of only density), while distributions with $\partial\Omega/\partial z\neq0$ are said *baroclinic* (pressure is not a function of only density).

Table 1. List of commonly used symbols.

$A_{pR} \equiv (\partial p_0/\partial R)/p_0$	Inverse of the pressure scale-length
$A_{pz} \equiv (\partial p_0/\partial z)/p_0$	Inverse of the pressure scale-height
$A_{\rho R} \equiv (\partial \rho_0/\partial R)/\rho_0$	Inverse of the density scale-length
$A_{\rho z} \equiv (\partial \rho_0/\partial z)/\rho_0$	Inverse of the density scale-height
$c_0 \equiv (p_0/\rho_0)^{1/2}$, $\tilde{c}_0 \equiv c_0/\Omega R$	Isothermal sound speed, normalized isothermal sound speed
$\mathcal{D} \equiv (k_R/k_z)\partial/\partial z - \partial/\partial R$	Derivative along surfaces of constant wave-phase
$\mathbf{k} = (k_R, k_z, k_\phi)$	Perturbation wave-vector
$k = \lvert\mathbf{k}\rvert = (k_R^2 + k_z^2 + k_\phi^2)^{1/2}$	Perturbation wave-number
$\tilde{k}_R \equiv k_R/k_\phi$, $\tilde{k}_z \equiv k_z/k_\phi$, $\tilde{k} \equiv k/k_\phi$	Normalized wave-vector components and wave-number
$n \equiv -\mathrm{i}\hat{\omega}$, $\tilde{n} \equiv n/\lvert\omega_\mathrm{d}\rvert$	Doppler-shifted perturbation frequency
p	Pressure perturbation throughout the paper (pressure in Section 2)
$p_0 = p_0(R, z)$	Unperturbed pressure
$s_0 \equiv \ln p_0 \rho_0^{-\gamma}$	Unperturbed specific entropy
T	Temperature perturbation throughout the paper (temperature in Section 2)
$T_0 = T_0(R, z)$	Unperturbed temperature
$\mathbf{v}_0 = \mathbf{v}_0(R, z) = (v_{0R}, v_{0z}, v_{0\phi})$	Unperturbed velocity
v_R, v_z, v_ϕ	Components of the velocity perturbation throughout the paper (of the velocity in Section 2)
$\tilde{v}_R \equiv v_R/\Omega R$, $\tilde{v}_z \equiv v_z/\Omega R$	Normalized components of the velocity perturbation
$\gamma = \frac{5}{3}$	Ratio of principal specific heats
$\gamma' \equiv \mathrm{d}\ln p_0/\mathrm{d}\ln \rho_0$	Local polytropic index of barotropic distributions.
$\Gamma_{pR} \equiv \partial \ln p_0/\partial \ln R$, $\Gamma_{pz} \equiv (R/z)\partial \ln p_0/\partial \ln \lvert z\rvert$	Local R and z logarithmic slopes of the unperturbed pressure
$\Gamma_{\rho R} \equiv \partial \ln \rho_0/\partial \ln R$, $\Gamma_{\rho z} \equiv (R/z)\partial \ln \rho_0/\partial \ln \lvert z\rvert$	Local R and z logarithmic slopes of the unperturbed density
$\Gamma_{\Omega R} \equiv \partial \ln \Omega/\partial \ln R$, $\Gamma_{\Omega z} \equiv (R/z)\partial \ln \Omega/\partial \ln \lvert z\rvert$	Local R and z logarithmic slopes of the angular velocity
κ	Thermal conductivity
Λ	Radiative cooling function
μ	Mean gas particle mass in units of the proton mass m_p
ρ	Density perturbation throughout the paper (density in Section 2)
$\rho_0 = \rho_0(R, z)$	Unperturbed density
$\tilde{\rho} \equiv \rho/\rho_0$	Normalized density perturbation
$\tau \equiv t\Omega$	Normalized time
ω	Perturbation frequency
$\hat{\omega} \equiv \omega - \mathbf{k}\cdot\mathbf{v}_0$	Doppler-shifted perturbation frequency
ω_BV, $\tilde{\omega}_\mathrm{BV} \equiv \omega_\mathrm{BV}/\lvert\omega_\mathrm{d}\rvert$	Brunt-Väisälä (or buoyancy) frequency (equations 18 and 21)
ω_rot, $\tilde{\omega}_\mathrm{rot} \equiv \omega_\mathrm{rot}/\lvert\omega_\mathrm{d}\rvert$	Characteristic frequency associated with rotation (equations 17 and 33)
ω_th, $\tilde{\omega}_\mathrm{th} \equiv \omega_\mathrm{th}/\Omega$	Thermal-instability frequency (equation 15)
ω_c, $\tilde{\omega}_\mathrm{c} \equiv (\omega_\mathrm{c} k_\phi^2)/(\Omega k^2)$	Thermal-conduction frequency (equation 14)
$\omega_\mathrm{d} \equiv \omega_\mathrm{th} + \omega_\mathrm{c}$, $\tilde{\omega}_\mathrm{d} \equiv \omega_\mathrm{d}/\Omega$	Characteristic frequency of dissipative processes
$\Omega = v_{0\phi}/R$	Angular velocity
$\Omega_R \equiv \partial(\Omega R)/\partial R$, $\Omega_z \equiv \partial(\Omega R)/\partial z$	Local R and z derivatives of $v_{0\phi} = \Omega R$

To address the question of the thermal stability of the rotating corona, we study the behaviour of the system in the presence of small (linear) thermal perturbations. These perturbations are not expected to have particular symmetries in a real system, so the relevant case is that of non-axisymmetric perturbations in the presence of differential rotation. It is well known that in a differentially rotating system such a problem is complicated by the effect of the shearing background on the perturbations (Cowling 1951; Goldreich & Lynden-Bell 1965; Lynden-Bell & Ostriker 1967). The analysis is much simpler in the case of axisymmetric perturbations or uniform rotation: though these assumptions are not expected to apply in general to galactic coronae, exploring these cases is useful to understand the behaviour of the more realistic, but far more complex case. Therefore, before addressing the full problem of non-axisymmetric perturbations in a differentially rotating corona, we will consider the cases of axisymmetric perturbations in a differentially rotating atmosphere and of non-axisymmetric perturbations under the assumption of uniform rotation.

3 THERMAL-STABILITY ANALYSIS: AXISYMMETRIC PERTURBATIONS

3.1 Derivation of the dispersion relation

Here the fluid is assumed to rotate differentially with $\Omega = \Omega(R, z)$ and that the perturbations are axisymmetric. Linearizing the hydrodynamic equations (4-8) with perturbations of the form $F_0 + F\exp(-\mathrm{i}\omega t + \mathrm{i}k_R R + \mathrm{i}k_z z)$ (where F_0 is the unperturbed quantity and $\lvert F\rvert \ll \lvert F_0\rvert$), in the limit of short-wavelength, low-frequency perturbations we get

$$-\mathrm{i}\hat{\omega}\rho + \mathrm{i}k_R v_R \rho_0 + \mathrm{i}k_z v_z \rho_0 = 0, \tag{9}$$

$$-\mathrm{i}\hat{\omega} v_R \rho_0 = -\mathrm{i}k_R p + A_{pR} c_0^2 \rho + 2\Omega v_\phi \rho_0, \tag{10}$$

$$-\mathrm{i}\hat{\omega}v_z\rho_0 = -\mathrm{i}k_z p + A_{pz}c_0^2\rho, \tag{11}$$

$$-\mathrm{i}\hat{\omega}v_\phi\rho_0 + v_R\rho_0\Omega_R + v_z\rho_0\Omega_z = -\rho_0 v_R\Omega, \tag{12}$$

$$\frac{T_0}{T\gamma}\left[-\mathrm{i}\hat{\omega}\frac{p}{p_0} + \mathrm{i}\gamma\hat{\omega}\frac{\rho}{\rho_0} + v_R\left(A_{pR} - \gamma A_{\rho R}\right) + v_z\left(A_{pz} - \gamma A_{\rho z}\right)\right] = -(\omega_{\rm c} + \omega_{\rm th}), \tag{13}$$

where $\hat{\omega} \equiv \omega - \mathbf{k}\cdot\mathbf{v}_0 = \omega - (k_R v_{0R} + k_z v_{0z})$ is the Doppler-shifted frequency, $c_0^2 \equiv p_0/\rho_0$ is the isothermal sound speed squared, $A_{pR} \equiv (\partial p_0/\partial R)/p_0$ and $A_{pz} \equiv (\partial p_0/\partial z)/p_0$ are the inverse of the pressure scale-length and scale-height, respectively. The following frequencies have also been defined: $\Omega_R \equiv \partial(\Omega R)/\partial R$, $\Omega_z \equiv \partial(\Omega R)/\partial z$, the thermal-conduction frequency

$$\omega_{\rm c} \equiv \left(\frac{\gamma-1}{\gamma}\right)\frac{k^2\kappa T_0^{7/2}}{p_0}, \tag{14}$$

and the thermal-instability frequency

$$\omega_{\rm th} \equiv -\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho_0^2\Lambda(T_0)}{p_0(\mu m_{\rm p})^2}\left[2 - \frac{\mathrm{d}\ln\Lambda(T_0)}{\mathrm{d}\ln T_0}\right]. \tag{15}$$

In terms of the defined quantities, the assumption of short-wavelength perturbations gives $|k_R|, |k_z| \gg |A_{\rho R}|, |A_{\rho z}|, |A_{pz}|, |A_{pR}|$, and $\Omega^2, \Omega_R^2, \Omega_z^2 \ll c_0^2k^2$, while the assumption of low-frequency perturbations gives $\omega^2 \ll c_0^2k^2$. In deriving the energy equation (13) we used $\rho_0 T \simeq -T_0\rho$, because it can be shown that in the considered limit $p/p_0 \ll \rho/\rho_0$ (see Appendix A). The system of equations (9-13) can be reduced to the following dispersion relation[1] for $n \equiv -\mathrm{i}\hat{\omega}$:

$$n^3 + n^2\omega_{\rm d} + (\omega_{\rm BV}^2 + \omega_{\rm rot}^2)n + \omega_{\rm rot}^2\omega_{\rm d} = 0, \tag{16}$$

where $\omega_{\rm d} \equiv \omega_{\rm th} + \omega_{\rm c}$ is the characteristic frequency of dissipative processes,

$$\omega_{\rm rot}^2 \equiv -\frac{k_z^2}{k^2}\frac{1}{R^3}\mathcal{D}(R^4\Omega^2) \tag{17}$$

is the differential rotation term,

$$\omega_{\rm BV}^2 \equiv -\frac{k_z^2}{k^2}\frac{\mathcal{D}p_0}{\rho_0\gamma}\mathcal{D}s_0 \tag{18}$$

is the buoyancy term, $s_0 \equiv \ln p_0\rho_0^{-\gamma}$ is the unperturbed specific entropy and, following Balbus (1995), we introduced the differential operator

$$\mathcal{D} \equiv \frac{k_R}{k_z}\frac{\partial}{\partial z} - \frac{\partial}{\partial R}, \tag{19}$$

which can be seen as taking derivatives along surfaces of constant wave phase. It may be useful to note that, by definition, $\omega_{\rm d}^2 \geq 0$, while $\omega_{\rm BV}^2$ and $\omega_{\rm rot}^2$ are not necessarily non-negative, and that n is defined so that the perturbation evolves in time as $\propto \exp(nt)$, thus the stable modes are those with $\mathrm{Re}(n) \leq 0$.

3.2 Limiting cases

Before analyzing the dispersion relation (16) in the general case, it is useful to discuss some limiting cases, which can be obtained when one or two among the three characteristic frequencies $\omega_{\rm BV}$, $\omega_{\rm rot}$ and $\omega_{\rm d}$ are zero. Let us start from the simplest cases, in which only one of these is non-null.

(i) *Case with* $\omega_{\rm d} = 0$ *and* $\omega_{\rm rot} = 0$. In this case there is no dissipation and the fluid is either non-rotating ($\Omega = 0$) or rotating differentially with vanishing gradient of the specific angular momentum [$\mathrm{d}(\Omega R^2)/\mathrm{d}R = 0$ when $\Omega = \Omega(R)$]. From equation (16) we have the dispersion relation

$$n^2 = -\omega_{\rm BV}^2, \tag{20}$$

so we have stability if the square of the Brunt-Väisälä frequency $\omega_{\rm BV}^2 > 0$. When $\Omega = \Omega(R)$ the Brunt-Väisälä frequency squared con be written

$$\omega_{\rm BV}^2 = \frac{k_z^2}{k^2}\frac{c_0^2A_{pz}^2}{\gamma}\left(\frac{\gamma}{\gamma'} - 1\right)\left(\frac{k_R}{k_z} - \frac{A_{pR}}{A_{pz}}\right)^2, \tag{21}$$

where, using the fact that p_0 is a function of only ρ_0 (because the distribution is barotropic; see Section 2), we defined

[1] The same dispersion relation is obtained by working from the beginning in the Boussinesq approximation, i.e. neglecting the term $-\mathrm{i}\hat{\omega}\rho$ in equation (9) and the term $-\mathrm{i}\hat{\omega}p/p_0$ in equation (13).

$$\gamma' \equiv \frac{\mathrm{d}\ln p_0}{\mathrm{d}\ln \rho_0}, \tag{22}$$

which can be considered a local polytropic index. It follows that, independently of the value of k_R/k_z, the condition for convective stability is $\gamma' < \gamma$, i.e. Schwarzschild's criterion (e.g. Tassoul 1978).

(ii) *Case with* $\omega_{\rm BV} = 0$ *and* $\omega_{\rm rot} = 0$. Here we assume that the characteristic frequencies associated with rotation and buoyancy are null. In the barotropic case these conditions are met when $\gamma' = \gamma$ (i.e. the radial gradient of the specific entropy is zero) and $\mathrm{d}(\Omega R^2)/\mathrm{d}R = 0$ (i.e. the radial gradient of the specific angular momentum is zero). The dispersion relation is

$$n = -\omega_{\rm d}, \tag{23}$$

so we have thermal instability if $\omega_{\rm d} < 0$, which is just Field's instability criterion (Field 1965). From the definition of $\omega_{\rm d} \equiv \omega_{\rm th} + \omega_{\rm c}$, it is clear that the condition for thermal instability is that the growth rate of the thermal perturbation ($|\omega_{\rm th}|$) must be faster than conductive damping (we recall that $\omega_{\rm c} \geq 0$, while typically $\omega_{\rm th} < 0$). For fixed unperturbed gas temperature T_0 and pressure p_0, $\omega_{\rm c}$ increases for increasing perturbation wave-number k (equation 14), while $\omega_{\rm th}$ is independent of k (equation 15), so there is a critical perturbation wavelength such that $\omega_{\rm d} < 0$ for longer wavelengths and $\omega_{\rm d} > 0$ for shorter wavelengths.

(iii) *Case with* $\omega_{\rm BV} = 0$ *and* $\omega_{\rm d} = 0$. In the absence of buoyancy and dissipation we obtain the dispersion relation

$$n^2 = -\omega_{\rm rot}^2, \tag{24}$$

so the stability criterion is $\omega_{\rm rot}^2 > 0$. The value of $\omega_{\rm rot}^2$ depends on the ratio k_R/k_z: $\omega_{\rm rot}^2$ is positive for all values of k_R/k_z if and only if $\partial\Omega/\partial z = 0$ and $\mathrm{d}(\Omega R^2)/\mathrm{d}R > 0$, i.e. the specific angular momentum must increase outwards: this is just Rayleigh's stability criterion (Chandrasekhar 1961).

In the following three limiting cases of the dispersion relation (16) only one among $\omega_{\rm BV}$, $\omega_{\rm rot}$ and $\omega_{\rm d}$ is null.

(i) *Case with* $\omega_{\rm BV} = 0$. In the absence of buoyancy, but for $\omega_{\rm rot} \neq 0$ and $\omega_{\rm d} \neq 0$, the dispersion relation is

$$(n^2 + \omega_{\rm rot}^2)(\omega_{\rm d} + n) = 0, \tag{25}$$

which is just a combination of the criteria obtained in the points ii) and iii) above, so (when $\omega_{\rm BV} = 0$) the presence of a gradient of the specific angular momentum does not modify the thermal-instability criterion in an interesting way. Specifically, when $\omega_{\rm d} < 0$ the medium is thermally unstable, independent of the presence and properties of rotation, while rotation can destabilize an otherwise thermally stable medium ($\omega_{\rm d} > 0$) if $\omega_{\rm rot}^2 < 0$. Thus, the condition for stability is $\omega_{\rm d} > 0$ and $\omega_{\rm rot}^2 > 0$, which holds for all values of k_R/k_z if and only if $\partial\Omega/\partial z = 0$ and $\mathrm{d}(\Omega R^2)/\mathrm{d}R > 0$.

(ii) *Case with* $\omega_{\rm d} = 0$. In the absence of dissipation, but for $\omega_{\rm rot} \neq 0$ and $\omega_{\rm BV} \neq 0$, one obtains the dispersion relation

$$n^2 = -(\omega_{\rm BV}^2 + \omega_{\rm rot}^2), \tag{26}$$

which leads to the well-known convective stability criterion for a rotating, stratified fluid $\omega_{\rm BV}^2 + \omega_{\rm rot}^2 > 0$, showing the stabilizing effect of rotation against convection. The inequality $\omega_{\rm BV}^2 + \omega_{\rm rot}^2 > 0$ is verified for all values of k_R/k_z if and only if

$$-\frac{1}{\gamma\rho_0}\nabla p_0 \cdot \nabla s_0 + \frac{1}{R^3}\frac{\partial R^4\Omega^2}{\partial R} > 0 \tag{27}$$

and

$$-\frac{\partial p_0}{\partial z}\left(\frac{\partial R^4\Omega^2}{\partial R}\frac{\partial s_0}{\partial z} - \frac{\partial R^4\Omega^2}{\partial z}\frac{\partial s_0}{\partial R}\right) > 0, \tag{28}$$

which is the classical Solberg-Høiland criterion (see Solberg 1936; Høiland 1941; Goldreich & Schubert 1967; Tassoul 1978; Balbus 1995).

(iii) *Case with* $\omega_{\rm rot} = 0$. When the fluid does not rotate, or, more generally, has a vanishing gradient of the specific angular momentum [$\mathrm{d}(\Omega R^2)/\mathrm{d}R = 0$ in the barotropic case], but $\omega_{\rm d} \neq 0$ and $\omega_{\rm BV} \neq 0$, the dispersion relation is

$$n^2 + \omega_{\rm d} n + \omega_{\rm BV}^2 = 0. \tag{29}$$

If the system is spherically symmetric, equation (29) reduces to the dispersion relation derived in MRB (see also BNF). When $\omega_{\rm d} > 0$ we have stability (damping by thermal conduction) if $\omega_{\rm BV}^2 > 0$, while we have convective instability if $\omega_{\rm BV}^2 < 0$. When $\omega_{\rm d} < 0$ we have overstability if $\tilde{\omega}_{\rm BV}^2 > \frac{1}{4}$, thermal instability if $0 < \tilde{\omega}_{\rm BV}^2 < \frac{1}{4}$ and convective instability if $\tilde{\omega}_{\rm BV}^2 < 0$.

3.3 Analysis of the general form of the dispersion relation

Here we consider the dispersion relation (16) in the general case in which all coefficients are non-null. Dividing by $|\omega_{\rm d}|^3$ we get

$$\tilde{n}^3 + \tilde{n}^2 + (\tilde{\omega}_{\rm BV}^2 + \tilde{\omega}_{\rm rot}^2)\tilde{n} + \tilde{\omega}_{\rm rot}^2 = 0, \qquad \text{if} \qquad \omega_{\rm d} > 0, \tag{30}$$

and

$$\tilde{n}^3 - \tilde{n}^2 + (\tilde{\omega}_{\rm BV}^2 + \tilde{\omega}_{\rm rot}^2)\tilde{n} - \tilde{\omega}_{\rm rot}^2 = 0, \qquad \text{if} \qquad \omega_{\rm d} < 0, \tag{31}$$

where the dimensionless quantities $\tilde{n} \equiv n/|\omega_{\rm d}|$, $\tilde{\omega}_{\rm BV}^2 \equiv \omega_{\rm BV}^2/\omega_{\rm d}^2$ and $\tilde{\omega}_{\rm rot}^2 \equiv \omega_{\rm rot}^2/\omega_{\rm d}^2$ have been introduced. In both cases the discriminant of the cubic equation is

$$\Delta = -27x^2 + (36y + 4)x - 4y(y^2 + 2y + 1), \tag{32}$$

where $x = \tilde{\omega}_{\rm BV}^2$ and $y = \tilde{\omega}_{\rm BV}^2 + \tilde{\omega}_{\rm rot}^2$.

Let us discuss first the case $\omega_{\rm d} > 0$ (equation 30 and diagram in the left-hand panel of Fig. 1). Applying the Routh-Hurwitz theorem (see Appendix B) we have that the real parts of all roots are negative (stability) if and only if $\tilde{\omega}_{\rm BV}^2 > 0$ and $\tilde{\omega}_{\rm rot}^2 > 0$. Thus the first quadrant of the diagram in the left-hand panel of Fig. 1 is a locus of stable configurations. Configurations in the other three quadrants can be either unstable or overstable. The blue curves in the diagrams correspond to discriminant $\Delta = 0$. In the bottom-left area defined by these curves $\Delta > 0$, thus we have three real roots (at least one negative), so the corresponding configurations are unstable. In the other regions $\Delta < 0$, so we have one real root ($\tilde{n}_1$) and two complex conjugate roots ($\tilde{n}_2$ and $\tilde{n}_3$). The sign of the real root $\tilde{n}_1$ can be determined because we know that $\tilde{n}_1\tilde{n}_2\tilde{n}_3 = -\tilde{\omega}_{\rm rot}^2$ (see Appendix B), and $\tilde{n}_2\tilde{n}_3$ is obviously positive. Thus we have instability in the upper-left region and overstability in the bottom-right region. In summary, the condition for stability against axisymmetric perturbations is $\omega_{\rm BV}^2 > 0$ and $\omega_{\rm rot}^2 > 0$, which holds for all values of k_R/k_z if $\partial\Omega/\partial z = 0$ [thus $\Omega = \Omega(R)$ and $p_0 = p_0(\rho_0)$], ${\rm d}(\Omega R^2)/{\rm d}R > 0$ and ${\rm d}\ln p_0/{\rm d}\ln\rho_0 < \gamma$. We note that the analysis of the case $\omega_{\rm d} > 0$ was carried out also by Lifschitz & Lebovitz (1993) in the context of the study of the stability of rotating stars in the presence of radiative diffusion (see also Goldreich & Schubert 1967; Sung 1974b, 1975; Balbus 2001).

Let us move to the case $\omega_{\rm d} < 0$ (equation 31 and diagram in the right-hand panel of Fig. 1). The same line of reasoning as for the case $\omega_{\rm d} > 0$ leads to the following conclusions. Given that the coefficient of $\tilde{n}^2$ is negative, we know from the Routh-Hurwitz theorem that in no case all the real parts of the roots are negative. In other words, all configurations will be either unstable or overstable. The blue curves in the diagram in the right-hand panel of Fig. 1 correspond to discriminant $\Delta = 0$. In the bottom-left area defined by these curves $\Delta > 0$, thus we have three real roots (at least one negative), so the corresponding configurations are unstable. In the other regions $\Delta < 0$, so we have one real root and two complex conjugate roots: in this case the sign of the real root is the same as the sign of $\tilde{\omega}_{\rm rot}^2$. Thus we have overstability in the upper-left region and instability elsewhere. It follows that a necessary condition for overstability is $\omega_{\rm BV}^2 > 0$ and $\omega_{\rm rot}^2 < 0$, which holds for all values of k_R/k_z if $\partial\Omega/\partial z = 0$ [thus $\Omega = \Omega(R)$ and $p_0 = p_0(\rho_0)$], ${\rm d}(\Omega R^2)/{\rm d}R < 0$ and $\gamma' < \gamma$. Note that this condition is not sufficient, because for positive, but small enough values of $\omega_{\rm BV}^2$ instability replaces overstability even if $\omega_{\rm rot}^2 < 0$ (see top-left quadrant of the right-hand panel of Fig. 1). The non-rotating case (briefly discussed at the end Section 3.2) is obtained from the diagram at $\tilde{\omega}_{\rm rot}^2 = 0$ (overstability if $\tilde{\omega}_{\rm BV}^2 > \frac{1}{4}$, and instability at lower values of $\tilde{\omega}_{\rm BV}^2$; see MRB).

3.4 Discussion

The above results indicate that the signs of $\omega_{\rm BV}^2$ and $\omega_{\rm rot}^2$ are crucial for the thermal-stability properties of a configuration. Both $\omega_{\rm BV}^2$ and $\omega_{\rm rot}^2$ depend not only on the properties of the unperturbed distribution, but also on the wave-vector of the perturbations through the coefficient k_z^2/k^2, and through k_R/k_z, which appears in the differential operator $\mathcal{D}$ defined in equation (19). A general result, valid for both positive and negative $\omega_{\rm d}$, is that if Ω depends on z the system is unstable, in the sense that it is always possible to choose k_R/k_z such that we are in the instability region of parameters. To look for stable or overstable configurations we must consider barotropic distributions, characterized by $\Omega = \Omega(R)$ and $p_0 = p_0(\rho_0)$. In this case $\omega_{\rm rot}^2$ is given by

$$\omega_{\rm rot}^2 = \frac{k_z^2}{k^2}\frac{1}{R^3}\frac{\rm d}{{\rm d}R}(R^4\Omega^2), \tag{33}$$

so it reduces to $(k_z/k)^2$ times Rayleigh's discriminant (Chandrasekhar 1961). The condition for $\omega_{\rm rot}^2 > 0$ is

$$\frac{{\rm d}\ln\Omega}{{\rm d}\ln R} > -2, \tag{34}$$

which is Rayleigh's stability criterion, requiring that the specific angular momentum must increase outwards. In addition, for barotropes the Brunt-Väisälä frequency is given by equation (21), so the condition for $\omega_{\rm BV}^2 > 0$ is Schwarzschild's criterion

$$\gamma' < \gamma. \tag{35}$$

In most applications we are interested in the case of convectively stable systems (satisfying condition 35) with outward-increasing specific angular momentum (satisfying condition 34). Coming back to the diagrams in Fig. 1 we conclude that

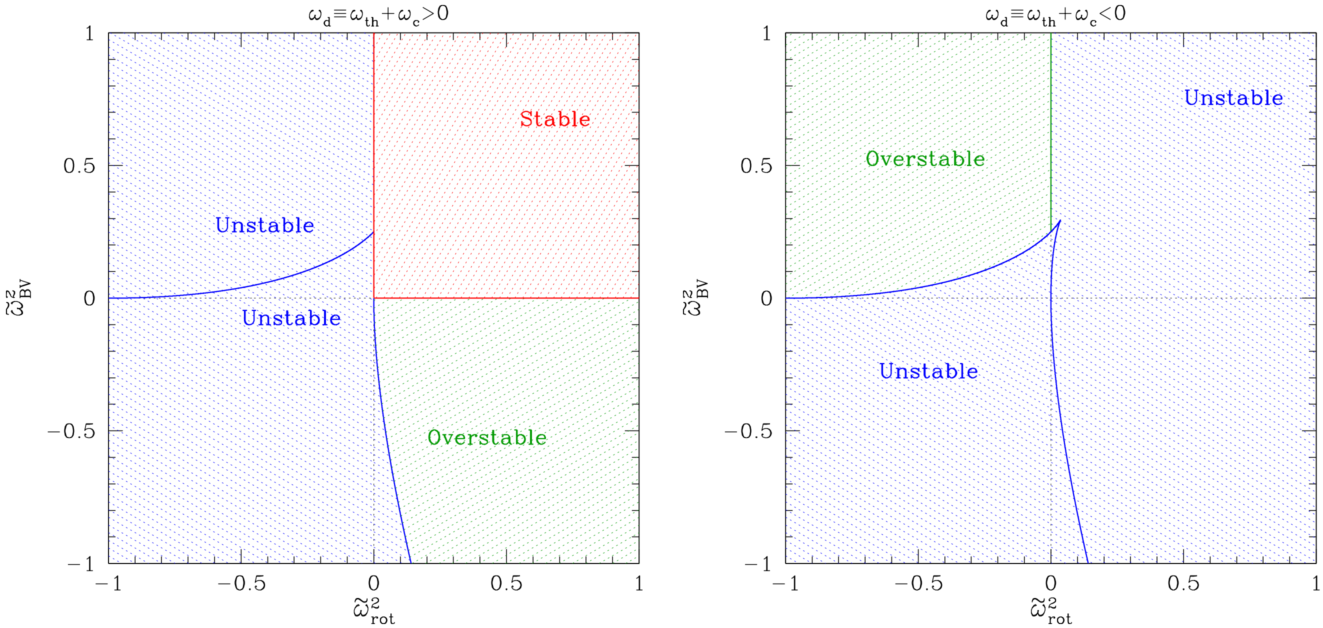

Figure 1. Domain of stability (red), overstability (green) and instability (blue) against axisymmetric perturbations in the plane $\tilde{\omega}^2_{\rm BV} \equiv \omega^2_{\rm BV}/|\omega_{\rm d}|^2$ versus $\tilde{\omega}^2_{\rm rot} \equiv \omega^2_{\rm rot}/|\omega_{\rm d}|^2$, for a differentially rotating, stratified fluid, when $\omega_{\rm d} > 0$ (left-hand panel) and $\omega_{\rm d} < 0$ (right-hand panel). $\omega^2_{\rm rot}$ and $\omega^2_{\rm BV}$ are given, respectively by equations (17) and (18).

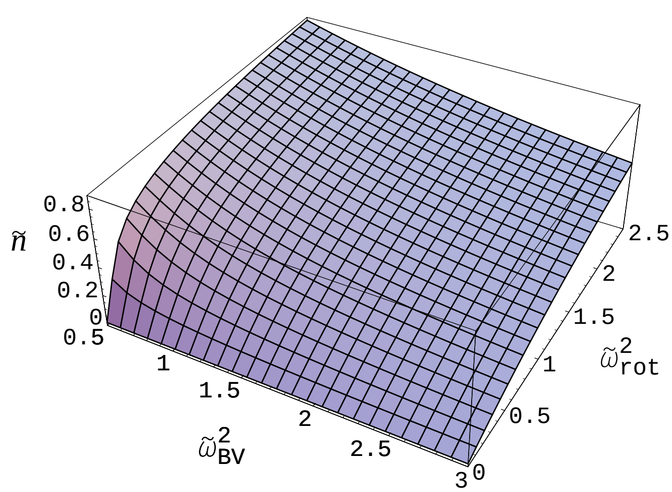

Figure 2. Growth rate $\tilde{n}$ of axisymmetric perturbations as a function of $\tilde{\omega}^2_{\rm rot}$ and $\tilde{\omega}^2_{\rm BV}$ for the unstable mode in the first quadrant ($\omega^2_{\rm BV} > 0$, $\omega^2_{\rm rot} > 0$) of the right-hand panel ($\omega_{\rm d} < 0$) of Fig. 1.

in the presence of cooling and thermal conduction, and assuming axisymmetric perturbations, these systems are stable if $\omega_{\rm d} > 0$ (damping by thermal conduction), while are unstable if $\omega_{\rm d} < 0$, independent of the value of the ratio $\omega^2_{\rm BV}/\omega^2_{\rm d}$ and on of the value of $\omega^2_{\rm rot} > 0$. This means that in the case $\omega_{\rm d} < 0$ (i.e., when the perturbation wavelength is short enough that the thermal-instability frequency is higher than the conductive-damping frequency) even arbitrarily slow rotation changes qualitatively the behaviour of the system transforming overstability (found in the absence of rotation; MRB) into instability. Physically, this different behaviour is due to the fact that, when the perturbations are axisymmetric, conservation of angular momentum tends to inhibit convection (Cowling 1951) and thus to prevent overdense regions from falling inward far enough to oscillate around the radius proper to their particular value of specific entropy. Note that the transition between overstability and instability is not abrupt, because the growth rate of the unstable mode goes to zero when $\omega^2_{\rm rot} \to 0$ (for $\tilde{\omega}^2_{\rm BV} > \frac{1}{4}$). This is apparent from Fig. 2, plotting the instability growth rate $\tilde{n}$, which is the real root obtained by solving equation (31) for

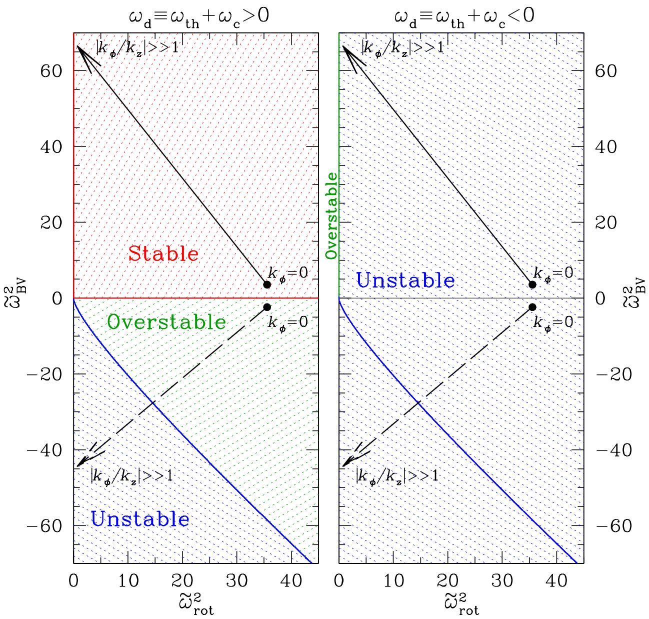

Figure 3. Domain of stability (red), overstability (green) and instability (blue) against non-axisymmetric perturbations in the plane $\tilde{\omega}^2_{\rm BV} \equiv \omega^2_{\rm BV}/|\omega_{\rm d}|^2$ (equation 45) versus $\tilde{\omega}^2_{\rm rot} = \omega^2_{\rm rot}/|\omega_{\rm d}|^2$ (equation 46), for a uniformly rotating, stratified gas, when $\omega_{\rm d} > 0$ (left-hand panel) and when $\omega_{\rm d} < 0$ (right-hand panel). The arrows show the effect of increasing the azimuthal wave-number $k_\phi = m/R$ from $k_\phi = 0$ (filled circles) to $|k_\phi/k_z| \gg 1$, for representative cases with $\gamma' = 1$ (solid lines) and $\gamma' = 3$ (long dashed line), assuming $k_z/k_R = 2$, $|\omega_{\rm d}|/\Omega = 0.3$, $\tilde{c}_0 = 2$, $\Gamma_{pz} = -1.5$, and $\Gamma_{pR} = -1.25$.

$\tilde{\omega}^2_{\rm rot} > 0$ and $\tilde{\omega}^2_{\rm BV} > 0$, as a function of $\tilde{\omega}^2_{\rm rot}$ and $\tilde{\omega}^2_{\rm BV}$. The growth rate, at fixed $\omega^2_{\rm BV}$, increases gradually from 0 for $\tilde{\omega}^2_{\rm rot}$ increasing from 0; the larger $\tilde{\omega}^2_{\rm BV}$ the larger must be $\tilde{\omega}^2_{\rm rot}$ to give substantial growth rates.

4 THERMAL-STABILITY ANALYSIS: NON-AXISYMMETRIC PERTURBATIONS

In the present Section we move to the next step of the stability analysis, which is the study of non-axisymmetric perturbations. Formally, it would be sufficient to perform the non-axisymmetric perturbation analysis for those configurations that are stable or overstable against non-axisymmetric perturbations, which might turn out to be unstable if more general perturbations are considered. Nevertheless, we will perform the non-axisymmetric perturbation analysis also for configurations found unstable against non-axisymmetric perturbations, because it is interesting to verify whether axisymmetry is a necessary condition for a perturbation to grow.

There are reasons to expect that non-axisymmetric thermal perturbations behave differently from axisymmetric ones. Cowling (1951) showed that rotation has a stabilizing effect against convection for axisymmetric perturbations (a consequence of angular momentum conservation), but not necessarily for non-axisymmetric perturbations: in the latter case the stabilizing effect of rotation is less and it vanishes for specific perturbations. This suggests that rotating galactic coronae (in the absence of heat conduction) might be thermally unstable against axisymmetric perturbations, but not against non-axisymmetric perturbations, because buoyancy should prevail, provided the gas entropy increases outwards (see also Nipoti 2010).

We now address quantitatively the problem of stability against non-axisymmetric perturbations. As mentioned above, in this case the analysis is rather complicated if the corona rotates differentially (Goldreich & Lynden-Bell 1965). Therefore, we first try to get an estimate of the effect of the deviation from axisymmetry of the perturbations by considering the simpler case of non-axisymmetric perturbations in a uniformly rotating corona (Section 4.1), and we defer the treatment of the differentially rotating case to Section 4.2.

4.1 Uniform rotation

Linearizing the hydrodynamic equations (4-8) with perturbations of the form $F_0 + F \exp(-{\rm i}\omega t + {\rm i}k_R R + {\rm i}k_z z + {\rm i}m\phi)$, where F_0 is the unperturbed quantity and $|F| \ll |F_0|$, and assuming angular velocity Ω independent of position, we get:

$$-{\rm i}\hat{\omega}\rho + {\rm i}k_R v_R \rho_0 + {\rm i}k_z v_z \rho_0 + {\rm i}k_\phi v_\phi \rho_0 = 0, \tag{36}$$

$$-\mathrm{i}\hat{\omega}v_R\rho_0 = -\mathrm{i}k_R p + A_{pR}c_0^2\rho + 2\Omega v_\phi\rho_0, \tag{37}$$

$$-\mathrm{i}\hat{\omega}v_z\rho_0 = -\mathrm{i}k_z p + A_{pz}c_0^2\rho, \tag{38}$$

$$-\mathrm{i}\hat{\omega}v_\phi\rho_0 + 2v_R\rho_0\Omega = -\mathrm{i}k_\phi p, \tag{39}$$

$$\frac{T_0}{T\gamma}\left[-\mathrm{i}\hat{\omega}\frac{p}{p_0} + \mathrm{i}\gamma\hat{\omega}\frac{\rho}{\rho_0} + v_R\left(A_{pR}-\gamma A_{\rho R}\right) + v_z\left(A_{pz}-\gamma A_{\rho z}\right)\right] = -\omega_{\mathrm{d}}, \tag{40}$$

where now $k^2 = k_R^2+k_z^2+k_\phi^2$ and $\hat{\omega} \equiv \omega - \mathbf{k}\cdot\mathbf{v}_0 = \omega - (k_R v_{0R} + k_z v_{0z} + k_\phi v_{0\phi})$, with $k_\phi \equiv m/R$. In deriving the equations above we have assumed, as done in Section 3, short wavelengths ($|k_\phi|, |k_R|, |k_z| \gg |A_{\rho R}|, |A_{\rho z}|, |A_{pR}|, |A_{pz}|, \Omega/c_0$), low frequencies ($\omega^2 \ll c_0^2k^2$) and $\rho_0 T = -T_0\rho$. The system of equations can be reduced to the dispersion relation

$$n^3 + n^2\omega_{\mathrm{d}} + (\omega_{\mathrm{BV}}^2 + \omega_{\mathrm{rot}}^2)n + \omega_{\mathrm{rot}}^2\omega_{\mathrm{d}} = 0, \tag{41}$$

where now

$$\omega_{\mathrm{BV}}^2 \equiv -\frac{1}{\gamma\rho_0}\frac{k_z^2}{k^2}\left[\mathcal{D}p_0\mathcal{D}s_0 + \frac{k_\phi^2}{k_z^2}\left(\frac{\partial p_0}{\partial R}\frac{\partial s_0}{\partial R} + \frac{\partial p_0}{\partial z}\frac{\partial s_0}{\partial z}\right)\right] = \frac{k_z^2}{k^2}\frac{c_0^2A_{pz}^2}{\gamma}\left(\frac{\gamma}{\gamma'}-1\right)\left[\left(\frac{k_R}{k_z}-\frac{A_{pR}}{A_{pz}}\right)^2 + \frac{k_\phi^2}{k_z^2}\left(1+\frac{A_{pR}^2}{A_{pz}^2}\right)\right] \tag{42}$$

is the buoyancy term, which for $k_\phi = 0$ reduces to the Brunt-Väisälä frequency in equation (21), with γ' given by equation (22), and

$$\omega_{\mathrm{rot}}^2 \equiv \frac{k_z^2}{k^2}4\Omega^2 \tag{43}$$

is the rotation term, which is just ω_{rot}^2 as defined in equation (17) for position-independent Ω. Note that in deriving the dispersion relation (41) we exploited the relation

$$\frac{\partial p_0}{\partial z}\frac{\partial s_0}{\partial R} = \frac{\partial p_0}{\partial R}\frac{\partial s_0}{\partial z}, \tag{44}$$

which holds because the surfaces of constant pressure and constant specific entropy coincide in fluids rotating with Ω independent of z (i.e. with barotropic distributions; see Section 2).

The dispersion relation (41) for non-axisymmetric perturbations in a uniformly rotating medium is formally identical to the dispersion relation (16) for axisymmetric perturbations. We just note that now ω_{BV}^2 depends on k_ϕ (equation 42) and $\omega_{\mathrm{rot}}^2 \geq 0$ (equation 43). When $\omega_{\mathrm{d}} = 0$ we obtain the dispersion relation $n^2 = -(\omega_{\mathrm{BV}}^2 + \omega_{\mathrm{rot}}^2)$ (with ω_{BV}^2 and ω_{rot}^2 given by equations 42 and 43), so the necessary and sufficient condition for convective stability against non-axisymmetric perturbations is $\omega_{\mathrm{BV}}^2 + \omega_{\mathrm{rot}}^2 > 0$, showing the stabilizing effect of uniform rotation against convection (in accordance with previous studies; Cowling 1951; Sung 1974a, 1975; Ryu & Goodman 1992). When $\omega_{\mathrm{d}} \neq 0$, the analysis of the dispersion relation (41) is the same as that of Section 3 and leads to the diagrams reported in Fig. 3, where now

$$\tilde{\omega}_{\mathrm{BV}}^2 \equiv \frac{\omega_{\mathrm{BV}}^2}{\omega_{\mathrm{d}}^2} = \frac{k_z^2}{k^2}\frac{\tilde{c}_0^2\Gamma_{pz}^2}{\gamma\tilde{\omega}_{\mathrm{d}}^2}\left(\frac{\gamma}{\gamma'}-1\right)\left[\left(\frac{k_R}{k_z}-\frac{\Gamma_{pR}}{\Gamma_{pz}}\right)^2 + \frac{k_\phi^2}{k_z^2}\left(1+\frac{\Gamma_{pR}^2}{\Gamma_{pz}^2}\right)\right] \tag{45}$$

and

$$\tilde{\omega}_{\mathrm{rot}}^2 \equiv \frac{\omega_{\mathrm{rot}}^2}{\omega_{\mathrm{d}}^2} = \frac{4}{\tilde{\omega}_{\mathrm{d}}^2}\frac{k_z^2}{k^2}, \tag{46}$$

having introduced the dimensionless quantities $\tilde{c}_0 \equiv c_0/\Omega R$, $\tilde{\omega}_{\mathrm{d}} \equiv \omega_{\mathrm{d}}/\Omega$,

$$\Gamma_{pR} \equiv \frac{\partial \ln p_0}{\partial \ln R} = RA_{pR} \quad \text{and} \quad \Gamma_{pz} \equiv \frac{R}{z}\frac{\partial \ln p_0}{\partial \ln |z|} = RA_{pz}. \tag{47}$$

For both $\omega_{\mathrm{d}} > 0$ (left-hand panel in Fig. 3) and $\omega_{\mathrm{d}} < 0$ (right-hand panel in Fig. 3) the domain of stability, instability and overstability are the same as in the first and fourth quadrants of the corresponding diagrams of Fig. 1.

The deviation from axisymmetry of the perturbation is measured by the ratio k_ϕ^2/k^2, on which both ω_{rot}^2 and ω_{BV}^2 depend. When the properties of the unperturbed system (in practice, $\tilde{\omega}_{\mathrm{d}}$, $\tilde{c}_0$, Γ_{pR}, Γ_{pz} and γ') and the ratio of the R and z wavenumbers k_R/k_z are fixed, the position in the $\tilde{\omega}_{\mathrm{rot}}^2$-$\tilde{\omega}_{\mathrm{BV}}^2$ plane depends only on k_ϕ^2/k^2. The sign of ω_{BV}^2 depends only on γ' (equation 42), so the quadrant in which a system is located does not change by varying the ratio k_ϕ^2/k^2 at fixed γ'. Therefore, when $\omega_{\mathrm{d}} > 0$ *a uniformly rotating system that is thermally stable against axisymmetric perturbations is stable also against non-axisymmetric perturbations* (see also Sung 1975). Similarly, when $\omega_{\mathrm{d}} < 0$ *a uniformly rotating system that is thermally unstable against axisymmetric perturbations is unstable also against non-axisymmetric perturbations.*

In order to illustrate the effect of varying k_ϕ^2/k^2, in Fig. 3 we plot the results for representative cases with $\gamma' < \gamma$ (namely $\gamma' = 1$; solid arrows) and $\gamma' > \gamma$ (namely $\gamma' = 3$; long dashed arrows), assuming $k_z/k_R = 2$, $|\omega_{\mathrm{d}}|/\Omega = 0.3$, $\tilde{c}_0 = 2$, $\Gamma_{pz} = -1.5$, and $\Gamma_{pR} = -1.25$. Of course the specific behaviour of the system depends on the choice of the parameters, however the examples shown in Fig. 3 represent qualitatively the trends of typical models in the four quadrants. When all the other

parameters are fixed, the modulus of $\omega_{\rm BV}^2$ increases for increasing k_ϕ^2/k^2. Therefore, when the system is convectively stable ($\omega_{\rm BV}^2 > 0$) increasing k_ϕ (i.e., considering non-axisymmetric perturbations with larger m or shorter azimuthal wavelength) moves our system towards the top-left of the first quadrants of the diagrams in Fig. 3. It is interesting to note that when $|k_\phi/k_z| \to \infty$, the system can become overstable, because $\omega_{\rm rot}^2 \to 0$ (we recall that for $\omega_{\rm d} < 0$ the system is overstable if $\omega_{\rm rot}^2 = 0$ and $\tilde{\omega}_{\rm BV}^2 > \frac{1}{4}$). Systems with $\omega_{\rm BV}^2 < 0$ move, for increasing k_ϕ, towards the bottom-left part of the diagrams in Fig. 3: in these cases, non-axisymmetric effects cannot change the unstable nature of a system with $\omega_{\rm d} < 0$, but it is interesting to note that a system with $\omega_{\rm d} > 0$, overstable against axisymmetric perturbations, can be unstable against high-m non-axisymmetric modes.

4.2 Differential rotation

Here we consider the case of non-axisymmetric perturbations in a differentially rotating corona, which is complicated by the effect of the shear on the perturbations. A possibility would be to consider disturbances in the form $F_0 + F(R,z)\exp(-{\rm i}\omega t + {\rm i}m\phi)$, where F_0 is the unperturbed quantity and $|F| \ll |F_0|$ (see e.g. Cowling 1951; Lin & Shu 1964; Sung 1974a, 1975; Fujimoto 1987; Hanawa 1987; Bertin et al. 1989): linearizing the hydrodynamic equations with these perturbations leads in general to a system of partial differential equations. As an alternative, one can adopt shearing coordinates (Goldreich & Lynden-Bell 1965) to reduce the problem to a system of ordinary differential equations (see also Balbus & Hawley 1992; Brandenburg & Dintrans 2006; Balbus et al. 2009). Here we follow the latter approach, and in particular we adopt the formalism used by Balbus & Hawley (1992) in their study of non-axisymmetric perturbations in a differentially rotating magnetized disc.

Let us consider an unperturbed distribution that is a solution of equations (4-8) with vanishing time partial derivatives and $\mathbf{v}_0 = (0, 0, v_{0\phi})$, where $v_{0\phi} = \Omega(R,z)R$ (for simplicity we assume here $v_{0R} = v_{0z} = 0$). Following Goldreich & Lynden-Bell (1965) we perform the change of coordinates $\phi' = \phi - \Omega(R,z)t$, $R' = R$, $z' = z$ and $t' = t$. The partial derivatives are transformed as follows:

$$\frac{\partial}{\partial\phi} = \frac{\partial}{\partial\phi'}, \tag{48}$$

$$\frac{\partial}{\partial R} = \frac{\partial}{\partial R'} - t\frac{\partial\Omega}{\partial R}\frac{\partial}{\partial\phi'}, \tag{49}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} - t\frac{\partial\Omega}{\partial z}\frac{\partial}{\partial\phi'}, \tag{50}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \Omega(R,z)\frac{\partial}{\partial\phi}. \tag{51}$$

We perturb the system with small non-axisymmetric disturbances. In the primed coordinates the perturbation takes the form of a plane wave, so we can write the perturbed quantities as $F_0 + F(t')\exp({\rm i}k_R'R' + {\rm i}k_z'z' + {\rm i}m\phi')$, with k_R', k_z' and m constant. It is convenient to work in the Boussinesq approximation: linearizing equations (4-8) we get

$$k_R(t')v_R + k_z(t')v_z + k_\phi v_\phi = 0, \tag{52}$$

$$\frac{{\rm d}v_R}{{\rm d}t'}\rho_0 + {\rm i}k_R(t')p - A_{pR}c_0^2\rho - 2\Omega v_\phi\rho_0 = 0, \tag{53}$$

$$\frac{{\rm d}v_z}{{\rm d}t'}\rho_0 + {\rm i}k_z(t')p - A_{pz}c_0^2\rho = 0, \tag{54}$$

$$\frac{{\rm d}v_\phi}{{\rm d}t'}\rho_0 + v_R\rho_0(\Omega + \Omega_R) + v_z\rho_0\Omega_z + {\rm i}k_\phi p = 0, \tag{55}$$

$$-\gamma\frac{{\rm d}\rho}{{\rm d}t'}\frac{1}{\rho_0} + v_R\left(A_{pR} - \gamma A_{\rho R}\right) + v_z\left(A_{pz} - \gamma A_{\rho z}\right) - \gamma\frac{\rho}{\rho_0}[\omega_{\rm c}(t') + \omega_{\rm th}] = 0, \tag{56}$$

where $k_\phi \equiv m/R'$,

$$k_R(t') = k_R' - mt'\frac{\partial\Omega}{\partial R} \tag{57}$$

and

$$k_z(t') = k_z' - mt'\frac{\partial\Omega}{\partial z}. \tag{58}$$

Note that now $\omega_{\rm c}$, still defined as in equation (14), depends on time, because it depends on $k^2(t') = k_R(t')^2 + k_z(t')^2 + k_\phi^2$. As implementations of the Boussinesq approximation we neglected the term proportional to ${\rm d}\rho/{\rm d}t'$ in equation (52) and the term proportional to ${\rm d}p/{\rm d}t'$ in equation (56), and we assumed $\rho_0 T = -T_0\rho$.

The system of five ordinary differential equations (52-56) in the unknown v_R, v_z, v_ϕ, ρ and p can be simplified by eliminating v_ϕ and p. In particular, in the momentum equations, v_ϕ can be eliminated by using the mass conservation equation (and its time derivative):

$$v_\phi = -\frac{k_R(t')}{k_\phi} v_R - \frac{k_z(t')}{k_\phi} v_z, \tag{59}$$

$$\frac{\mathrm{d}v_\phi}{\mathrm{d}t'} = -\frac{k_R(t')}{k_\phi}\frac{\mathrm{d}v_R}{\mathrm{d}t'} - \frac{k_z(t')}{k_\phi}\frac{\mathrm{d}v_z}{\mathrm{d}t'}. \tag{60}$$

The pressure perturbation p can be eliminated by combining the momentum equations as follows: we subtract the z equation (multiplied by k_R) from the R equation (multiplied by k_z), and we subtract the z equation (multiplied by k_ϕ) from the ϕ equation (multiplied by k_z). After rearrangement and simplification, we end up with the following system of three coupled ordinary differential equations (in the variables v_R, v_z and ρ), which fully describes the evolution of the non-axisymmetric perturbations in a differentially rotating corona:

$$\frac{\mathrm{d}v_R}{\mathrm{d}t} = \frac{2k_R k_\phi \Omega}{k^2}\left(\frac{\partial \ln\Omega}{\partial \ln R} - \frac{k_z^2}{k_\phi^2}\right) v_R + \frac{2k_R k_\phi \Omega}{k^2}\left(\frac{R}{z}\frac{\partial \ln\Omega}{\partial \ln|z|} - \frac{k_z}{k_R}\frac{k_z^2+k_\phi^2}{k_\phi^2}\right) v_z + \frac{c_0^2 A_{pR}}{\rho_0}\left(\frac{k_z^2+k_\phi^2}{k^2} - \frac{A_{pz}}{A_{pR}}\frac{k_R k_z}{k^2}\right)\rho, \tag{61}$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{2k_z k_\phi \Omega}{k^2}\left(\frac{\partial \ln\Omega}{\partial \ln R} + \frac{k_R^2+k_\phi^2}{k_\phi^2}\right) v_R + \frac{2k_z k_\phi \Omega}{k^2}\left(\frac{R}{z}\frac{\partial \ln\Omega}{\partial \ln|z|} + \frac{k_z k_R}{k_\phi^2}\right) v_z + \frac{c_0^2 A_{pz}}{\rho_0}\left(\frac{k_R^2+k_\phi^2}{k^2} - \frac{A_{pR}}{A_{pz}}\frac{k_R k_z}{k^2}\right)\rho, \tag{62}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\rho_0}{\gamma}\left(A_{pR} - \gamma A_{\rho R}\right) v_R + \frac{\rho_0}{\gamma}\left(A_{pz} - \gamma A_{\rho z}\right) v_z - \left(\omega_\mathrm{c} + \omega_\mathrm{th}\right)\rho, \tag{63}$$

where we dropped the explicit time dependence of k_R, k_z and ω_c. It is useful to rewrite the system above more compactly using only dimensionless quantities:

$$\frac{\mathrm{d}\tilde{v}_R}{\mathrm{d}\tau} = \frac{2\tilde{k}_R}{\tilde{k}^2}\left(\Gamma_{\Omega R} - \tilde{k}_z^2\right)\tilde{v}_R + \frac{2\tilde{k}_R}{\tilde{k}^2}\left[\Gamma_{\Omega z} - \frac{\tilde{k}_z}{\tilde{k}_R}(\tilde{k}_z^2+1)\right]\tilde{v}_z + \tilde{c}_0^2\Gamma_{pR}\left(\frac{\tilde{k}_z^2+1}{\tilde{k}^2} - \frac{\Gamma_{pz}}{\Gamma_{pR}}\frac{\tilde{k}_R\tilde{k}_z}{\tilde{k}^2}\right)\tilde{\rho}, \tag{64}$$

$$\frac{\mathrm{d}\tilde{v}_z}{\mathrm{d}\tau} = \frac{2\tilde{k}_z}{\tilde{k}^2}\left(\Gamma_{\Omega R} + \tilde{k}_R^2 + 1\right)\tilde{v}_R + \frac{2\tilde{k}_z}{\tilde{k}^2}\left(\Gamma_{\Omega z} + \tilde{k}_z\tilde{k}_R\right)\tilde{v}_z + \tilde{c}_0^2\Gamma_{pz}\left(\frac{\tilde{k}_R^2+1}{\tilde{k}^2} - \frac{\Gamma_{pR}}{\Gamma_{pz}}\frac{\tilde{k}_R\tilde{k}_z}{\tilde{k}^2}\right)\tilde{\rho}, \tag{65}$$

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\tau} = \frac{1}{\gamma}\left(\Gamma_{pR} - \gamma\Gamma_{\rho R}\right)\tilde{v}_R + \frac{1}{\gamma}\left(\Gamma_{pz} - \gamma\Gamma_{\rho z}\right)\tilde{v}_z - \left(\tilde{\omega}_\mathrm{c}\tilde{k}^2 + \tilde{\omega}_\mathrm{th}\right)\tilde{\rho}, \tag{66}$$

where $\tilde{v}_R \equiv v_R/\Omega R$, $\tilde{v}_z \equiv v_z/\Omega R$, $\tilde{\rho} \equiv \rho/\rho_0$, $\tau \equiv t'\Omega$, $\tilde{k}_R \equiv k_R/k_\phi$, $\tilde{k}_z \equiv k_z/k_\phi$, $\tilde{k} \equiv k/k_\phi$,

$$\Gamma_{\rho R} \equiv \frac{\partial \ln\rho_0}{\partial \ln R} = R A_{\rho R}, \qquad \Gamma_{\rho z} \equiv \frac{R}{z}\frac{\partial \ln\rho_0}{\partial \ln|z|} = R A_{\rho z}, \tag{67}$$

$$\Gamma_{\Omega R} \equiv \frac{\partial \ln\Omega}{\partial \ln R}, \qquad \Gamma_{\Omega z} \equiv \frac{R}{z}\frac{\partial \ln\Omega}{\partial \ln|z|}, \tag{68}$$

Γ_{pR} and Γ_{pz} are defined in equations (47),

$$\tilde{\omega}_\mathrm{c} \equiv \frac{\omega_\mathrm{c} k_\phi^2}{\Omega k^2} = \frac{1}{\Omega}\left(\frac{\gamma-1}{\gamma}\right)\frac{k_\phi^2 \kappa T_0^{7/2}}{p_0}, \tag{69}$$

and

$$\tilde{\omega}_\mathrm{th} \equiv \frac{\omega_\mathrm{th}}{\Omega}. \tag{70}$$

In equations (64-66), besides the unknown $\tilde{v}_R(\tau)$, $\tilde{v}_z(\tau)$ and $\tilde{\rho}(\tau)$, the only other time-dependent quantities are

$$\tilde{k}_R(\tau) = \tilde{k}'_R - \Gamma_{\Omega R}\tau, \tag{71}$$

$$\tilde{k}_z(\tau) = \tilde{k}'_z - \Gamma_{\Omega z}\tau, \tag{72}$$

and $\tilde{k}^2(\tau) = \tilde{k}_R^2 + \tilde{k}_z^2 + 1$, where $\tilde{k}'_R \equiv k'_R/k_\phi$ and $\tilde{k}'_z \equiv k'_z/k_\phi$ (note that $\tilde{\omega}_\mathrm{c}$ does not depend on time). The system of ordinary differential equations is thus completed by specifying at $\tau = 0$ the values of $\tilde{v}_R$, $\tilde{v}_z$ and $\tilde{\rho}$, and by choosing the values of the following parameters: $\tilde{k}'_R$, $\tilde{k}'_z$, $\Gamma_{\Omega R}$, $\Gamma_{\Omega z}$, Γ_{pR}, Γ_{pz}, $\Gamma_{\rho R}$, $\Gamma_{\rho z}$, $\tilde{c}_0$, $\tilde{\omega}_\mathrm{c}$ and $\tilde{\omega}_\mathrm{th}$. A full exploration of the parameter space is prohibitive, so we will present solutions for specific relevant cases, obtained by numerically integrating the system of equations (64-66) with a fourth-order Runge-Kutta method.

4.2.1 Barotropic distributions

Let us focus first on solutions of the system of equations (64-66) in the case of barotropic distributions (see Section 2), for which $\Omega = \Omega(R)$ and $p_0 = p_0(\rho_0)$, with local polytropic index $\gamma' = \mathrm{d}\ln p_0/\mathrm{d}\ln\rho_0$. We are interested in estimating the effect of the non-axisymmetry of the perturbations and of the differential rotation on the thermal stability of the fluid, so it is convenient to make a comparison with the results obtained for axisymmetric perturbations (Section 3) and uniform rotation (Section 4.1). In those cases, we found it useful to explore separately the cases $\omega_\mathrm{d} = 0$, $\omega_\mathrm{d} > 0$ and $\omega_\mathrm{d} < 0$, where $\omega_\mathrm{d} = \omega_\mathrm{th} + \omega_\mathrm{c}$ is the characteristic frequency of dissipative processes. In the case of non-axisymmetric perturbations and differential rotation

$\omega_{\rm d} = \Omega(\tilde{\omega}_{\rm c}\tilde{k}^2 + \tilde{\omega}_{\rm th})$ depends on time through $\tilde{k}(t')$, so for comparison with the previous analysis it is convenient to distinguish the following cases:

(i) *Case with* $\tilde{\omega}_{\rm c} = 0$ *and* $\tilde{\omega}_{\rm th} = 0$. This is the non-dissipative case, which has been studied in previous work on convective instability in rotating stratified fluids (Cowling 1951; Sung 1975). In accordance with Sung (1975), we find that the combination of differential rotation and non-axisymmetry of the perturbations can have a destabilizing effect against convection. For instance, we find that—for sufficiently large values of k_ϕ—adiabatic ($\gamma' = \gamma$) configurations are stable for $\Gamma_{\Omega R} = 0$ (uniform rotation), but are unstable for $\Gamma_{\Omega R} \neq 0$ (differential rotation), even when the specific angular momentum increases outwards ($\Gamma_{\Omega R} > -2$).

(ii) *Case with* $\tilde{\omega}_{\rm c} > 0$ *and* $\tilde{\omega}_{\rm th} = 0$. This is a dissipative case, with positive dissipative term, analogous to the model with radiative diffusion studied by Sung (1975). For axisymmetric perturbations or uniform rotation, when $\omega_{\rm d} > 0$ the only stable configurations are those characterized by $\gamma' < \gamma$ and $d(\Omega R^2)/dR > 0$ (i.e. $\Gamma_{\Omega R} > -2$); the other configurations are unstable or overstable (see left-hand panels of Figs. 1 and 3). We calculated numerical solutions for configurations with $\gamma' < \gamma$ and $\Gamma_{\Omega R} > -2$ (known to be stable against axisymmetric perturbations), finding them stable also against non-axisymmetric perturbations for a wide range of values of $\tilde{k}_R$ and $\tilde{k}_z$ (see also Sung 1975).

(iii) *Case with* $\tilde{\omega}_{\rm c} = 0$ *and* $\tilde{\omega}_{\rm th} < 0$. This is a dissipative case, with negative dissipative term, to be compared with the cases of axisymmetric perturbations or uniform rotation with $\omega_{\rm d} < 0$ (right-hand panels of Figs. 1 and 3). In those cases, the systems were found unstable in the relevant parameter regime $\gamma' < \gamma$ and $\Gamma_{\Omega R} > -2$ (outward-increasing specific entropy and angular momentum). We explored the stability against non-axisymmetric perturbations for several configurations with $\gamma' < \gamma$ and $\Gamma_{\Omega R} > -2$ finding that *the combination of non-axisymmetry of the perturbations and differential rotation can turn instability into overstability*. This can be seen from Fig. 4, showing, for example, the time evolution of non-axisymmetric perturbations in three configurations (models 4a, 4b and 4c) differing only for the values of $\Gamma_{\Omega R}$ (measuring the degree of differential rotation), $\tilde{k}'_R$ and $\tilde{k}'_z$ (measuring the deviation from axisymmetry of the perturbations). The modes considered in models 4a and 4b have k_ϕ of the order of k_R and k_z, so they are relevant to blob-like perturbations. In model 4a rotation is uniform, and the system is unstable, consistent with the calculations of Section 4.1. Model 4b differs from model 4a only because differential rotation (with $\Gamma_{\Omega R} = -1$) replaces uniform rotation: as a consequence the system is overstable. It must be noted that the growth rate of the amplitude of the overstable mode shown in panel 4b is not negligible: the perturbation, which is assumed of the order of 1% at $\tau = 0$, becomes nonlinear at $\tau \sim 25$. Model 4c also rotates differentially with $\Gamma_{\Omega R} = -1$, as model 4b, but now the value of the azimuthal wave-number is smaller by a factor of 50 than in model 4b, meaning that the perturbation is nearly axisymmetric: this mode, which is relevant for disturbances very elongated along ϕ, is clearly unstable.

(iv) *Case with* $\tilde{\omega}_{\rm c} > 0$ *and* $\tilde{\omega}_{\rm th} < 0$. In this case both thermal conduction and cooling are present. Let us focus again on cases in which $\gamma' < \gamma$ and $\Gamma_{\Omega R} > -2$ (outward-increasing specific entropy and angular momentum). Considering the initial value of the dissipative frequency $\omega_{\rm d}(0) = \Omega(\tilde{\omega}_{\rm c}\tilde{k}'^2 + \tilde{\omega}_{\rm th})$, where $\tilde{k}'^2 \equiv \tilde{k}'^2_R + \tilde{k}'^2_z + 1 = \tilde{k}^2(0)$, we distinguish different cases according to the sign of $\omega_{\rm d}(0)$. When $\omega_{\rm d}(0) > 0$, all explored cases of non-axisymmetric perturbations in the presence of differential rotation turn out to be stable, so differential rotation and deviation from axisymmetry do not have destabilizing effects on configurations stabilized by thermal conduction. When $\omega_{\rm d}(0) < 0$ we find stability, overstability or instability depending on the values of the parameters. Given that in the limit of uniform rotation or axisymmetric perturbations we always have instability in this parameter regime (see right-hand panels of Figs. 1 and 3), it is clear that *non-axisymmetry of the perturbations and differential rotation can have a stabilizing effect.* An example is shown in Fig. 5, plotting the time evolution of non-axisymmetric perturbations in three configurations (5a, 5b and 5c), having the same values parameters as the corresponding configurations considered in Fig. 4, but $\tilde{\omega}_{\rm c}k'^2 = 0.1$, so that the initial value of the dissipative frequency is $\omega_{\rm d}(0) = -0.2\Omega$. The uniformly rotating case (model 5a) is unstable, consistent with the results of Section 4.1. Interestingly, the differentially rotating case with the same wave-vector (model 5b) is stable: in this case $\omega_{\rm d}(t)$ is negative at $t = 0$, but soon becomes positive because $\tilde{k}^2_R$ increases as a consequence of differential rotation (making also $\tilde{k}^2$ and $\omega_{\rm c}$ increase). We recall that the analogous configuration, in the absence of conduction is overstable (model 4b in Fig. 4). Physically, the shear tends to distort any overdense region by making it narrow in the R direction, so the growth of the perturbation is more easily damped by thermal conduction. As models 4a and 4b, also models 5a and 5b are characterized by modes with k_ϕ of the order of k_R and k_z, so they are of interest for blob-like perturbations. If the azimuthal wave-number is small enough (i.e. the mode is almost axisymmetric, relevant to overdensities very elongated along ϕ) the system appears unstable even in the presence of differential rotation (model 5c). Formally, also in this case $\omega_{\rm d}(t)$ becomes positive over sufficiently long times, and eventually the perturbation is damped. However, this must not necessarily happen in reality, because it is clear that the linear analysis breaks when at least one among $\tilde{v}_R$, $\tilde{v}_z$ and $\tilde{\rho}$ becomes much larger than unity.

4.2.2 Baroclinic distributions

In the previous Section we focused on barotropic distributions, i.e. configurations in which $\Omega = \Omega(R)$ and $p_0 = p_0(\rho_0)$. We consider here the problem of the stability against non-axisymmetric perturbations of systems with baroclinic distributions:

in other words, we allow for the fact that Ω can depend on z as well as on R (implying that p_0 is not stratified with ρ_0; see Section 2). In terms of the formalism here adopted, we describe solutions of equations (64-66) in cases with $\Gamma_{\Omega z} \neq 0$.

As it happens for the barotropic cases, also baroclinic configurations are found to be stable, unstable or overstable depending on the specific choice of the parameters. In general we find that adding a relatively small vertical gradient of Ω does not change the stability properties of a system. However, we found that in some cases the fact that Ω depends on z can have destabilizing effects. In order to isolate this phenomenon, we considered models with the same parameters as the barotropic models discussed above, but with $\Gamma_{\Omega z} \neq 0$. For small values of $|\Gamma_{\Omega z}|$ the baroclinic models behave like the corresponding barotropic models, but—in the absence of thermal conduction—instability replaces overstability for sufficiently large $|\Gamma_{\Omega z}|$ (i.e. sufficiently strong dependence on z of the angular velocity). An example of thermally unstable baroclinic configuration is model 4d in Fig. 4, which has the same values of the parameters as the (overstable) barotropic model 4b, but $\Gamma_{\Omega z} = -0.6$. When thermal conduction is present baroclinic models are found to be stable if the "corresponding" barotropic models are stable: compare, in Fig. 5, the barotropic model 5b with the baroclinic model 5d.

5 IMPLICATIONS FOR GALACTIC CORONAE

Here we use the results obtained above to address the question of whether the coronae of disc galaxies can fragment, via thermal instability, into cool, pressure-supported clouds, which, in the case of the Milky Way, would be identified with the high-velocity clouds. As already pointed out, the physical properties of the galactic coronae of disc galaxies are poorly constrained observationally. Such coronae are expected to be stratified, almost in equilibrium at the system's virial temperature, similar to the hot atmospheres of massive elliptical galaxies and galaxy clusters, but characterized by significantly lower gas temperature and density, and possibly by non-negligible rotation.

MRB studied the problem of the thermal stability of the hot atmospheres of galaxy clusters, using Eulerian plane-wave perturbations and assuming that the gas does not rotate, and concluded that these systems are not prone to significant thermal instability. It has been pointed out that when a background flow is present, the use of Lagrangian perturbations is preferable to that of Eulerian perturbations (Balbus 1988). However, the main results of MRB have been confirmed by the Lagrangian study of Balbus & Soker (1989). BNF applied the thermal-stability analysis of MRB to non-rotating models of the coronae of disc galaxies, finding that the result obtained for the highest-temperature atmospheres of clusters (thermal stability or overstability) apply also to these lower-temperature systems, unless the gas distribution is perfectly adiabatic (which is unexpected). In the present paper we tried to verify whether the conclusions drawn from the analysis of the non-rotating cases extend also to rotating coronae. As already stressed, for simplicity we treated the gas as unmagnetized, so our results apply to real systems only to the extent that their magnetic fields do not influence substantially the behaviour of thermal perturbations.

From the calculations reported in the above Sections it is apparent that rotation introduces some mathematical complexity in the stability analysis. As a consequence, it is hard to draw general conclusions about the thermal-stability properties of rotating coronae, which can be thermally stable, overstable or unstable, depending on the distribution of specific entropy and angular momentum, but also on the nature of the perturbations. If we knew the density, temperature and specific angular momentum distributions of the corona of the Milky Way (or of an external disc galaxy), the dispersion relations and the differential equations derived above could be used to determine unambiguously whether a given perturbation grows. Unfortunately, the uncertainties on the physical properties of the coronae are such that we cannot be conclusive about their thermal stability or instability. Nevertheless, some general considerations can be done.

First of all, our calculations show that the stabilizing effect of thermal conduction found in non-rotating systems occurs also in the presence of rotation (see cases with $\omega_{\rm d} > 0$ in Sections 3 and 4). As in the non-rotating case, *thermal perturbations with sufficiently short wavelength do not grow.* The interesting question is then whether perturbations with sufficiently long wavelength (i.e. small enough wave-number k, so that $\omega_{\rm d} < 0$) grow. It seems reasonable that galactic coronae are characterized by outward-increasing specific entropy[2] and angular momentum, so in the present discussion we can specialize to configurations with $\omega^2_{\rm BV} > 0$ and $\omega^2_{\rm rot} > 0$. In Section 3 we showed that when $\omega_{\rm d} < 0$, such configurations are *unstable* against axisymmetric perturbations, in contrast with analogous non-rotating configurations, which have been shown to be overstable. Taken at face value this result indicates that the presence of even slow rotation can modify qualitatively the stability properties of a corona: formally, we should conclude that rotating galactic coronae are thermally unstable, because there is at least one mode that grows (in fact, all the axisymmetric modes grow). However, such instability must not necessarily have important consequences in the practical problem of the thermal stability of galactic coronae. We note that the high-velocity clouds, which would be the end-products of the thermal instability of the Milky Way corona, do not appear axisymmetric with respect to the rotation

[2] BNF showed that, at least in the absence of rotation, very special conditions must be met in order to have a corona with flat specific entropy profile.

axis of the Galaxy. Moreover, axisymmetric structures of cold gas far from the plane are not observed, in general, in external disc galaxies.

It is then interesting to study the behaviour of non-axisymmetric perturbations, which are relevant to the problem of whether blob-like, cool clouds can condense out of the corona. We have shown in Section 4 that at least some differentially rotating configurations, which are unstable against axisymmetric perturbations, are overstable (model 4b in Fig. 4) or stable (model 5b in Fig. 5) against non-axisymmetric perturbations with sufficiently high azimuthal wave-number (specifically, to modes with k_ϕ of the order of k_R and k_z, which describe blob-like disturbances). It follows that an overdense, cool blob in a differentially rotating corona is unlikely to condense, even if the thermal-instability frequency is faster than conductive damping (i.e., $\omega_{\rm d} < 0$).

From our analysis it also emerged that the distribution of specif angular momentum is particularly important for the thermal stability of the coronae. For instance, uniform rotation (unexpected in real systems) would favour thermal instability, while (more realistic) differential rotation with outward-increasing specific angular momentum tends to stabilize. A vertical gradient of Ω (Ω decreasing for increasing z), might be expected if the kinematic properties of the coronae reflect those of the neutral extra-planar gas of disc galaxies (e.g. Fraternali 2009, and references therein). In the absence of thermal conduction such a vertical gradient, if sufficiently strong, might have a destabilizing effect (compare baroclinic and barotropic models in Fig. 4). However, such an instability is not expected to occur in the presence of even highly suppressed conductivity. In fact, the plots in the right-hand panel of Fig. 5 show clearly that conductive damping is enhanced by the shear induced by differential rotation: even non-axisymmetric perturbations that have initially long enough wavelength—such that $\omega_{\rm d}(0) < 0$—are effectively sheared and therefore stabilized by thermal conduction, independently of whether the distribution is barotropic (model 5b) or baroclinic (model 5d).

Altogether these considerations lead to the conclusion that, generally speaking, differential rotation does not make galactic coronae thermally unstable to non-axisymmetric perturbations. We cannot exclude that thermal perturbations grow in specific cases, but a definitive answer to the question of whether the corona of a given galaxy is thermally unstable can be obtained only when a detailed model of the corona, including its specific angular momentum distribution, is available.

6 SUMMARY AND CONCLUSIONS

Motivated by the question of whether cool, pressure-supported clouds can condense out of the hot coronae of disc galaxies, we studied the problem of the thermal stability of rotating stratified fluids, in the presence of radiative cooling and thermal conduction. As in the non-rotating case, the time evolution of a perturbation depends on its wavelength, so it is useful to distinguish short-wavelength perturbations (such that the dissipative frequency $\omega_{\rm d}$ is positive) from long-wavelength perturbations (such that the dissipative frequency $\omega_{\rm d}$ is negative). We found that—against either axisymmetric or non-axisymmetric perturbations—a *uniformly rotating*, convectively stable configuration is thermally stable when $\omega_{\rm d} > 0$ (damping by thermal conduction), but is thermally unstable when $\omega_{\rm d} < 0$. Similarly—against axisymmetric perturbations—a *differentially rotating*, convectively stable corona with outward-increasing specific angular momentum is thermally stable when $\omega_{\rm d} > 0$, but is thermally unstable when $\omega_{\rm d} < 0$. Non-axisymmetric perturbations in the presence of differential rotation behave differently from the axisymmetric ones. In the *absence of thermal conduction*, the combination of non-axisymmetry of the perturbation and of differential rotation has a stabilizing effect (turning instability into overstability), and barotropic distributions tend to be more stable than baroclinic distributions. In the *presence of thermal conduction*, stability replaces overstability: in differentially rotating systems, the shear makes conductive damping of non-axisymmetric disturbances particularly effective for both barotropic and baroclinic distributions.

These results have been discussed in the context of the problem of the growth of thermal perturbations in galactic coronae. The above calculations allow to address unambiguously this question when applied to a specific model of rotating galactic corona. Unfortunately, given that the physical properties of the hot atmospheres of disc galaxies (in particular, the distribution of specific entropy and angular momentum) are very poorly constrained observationally, it is difficult to be conclusive about whether these systems are prone to thermal stability. Additional uncertainties come from magnetic fields, which are expected to be present in real systems, but have been neglected for simplicity in the studied models. The question of the effect of magnetic fields on the thermal instability is very interesting and would require a full thermal-stability analysis of a rotating, magnetized corona, for which our calculations may be a starting point.

Though limited by the mentioned uncertainties, the present work gives further indications against the hypothesis that thermal instability is important for galactic coronae. In particular, though rotation can destabilize against thermal disturbances very elongated in the azimuthal direction, we argued that blob-like thermal perturbations are unlikely to grow in a differentially rotating corona. This finding, combined with previous results on non-rotating coronae (BNF), suggests that the high-velocity clouds of the Milky Way did not form spontaneously from small thermal perturbations in the Galactic corona, but must be of external origin. A possibility is that at least the seeds of these clouds are formed originally as a consequence of stripping of gas-rich satellites or cosmic infall of cold gas. These cool gaseous seeds might then grow while travelling through the hot

Galactic corona (Sommer-Larsen 2006; Kereš & Hernquist 2009), for instance via turbulent mixing (Marinacci et al. 2010). Provided that the overdensities associated with these accreted seeds are substantial (i.e. they are non-linear perturbations), such a scenario for the formation of the high-velocity clouds is not in contrast with the results of the present paper, in which only linear perturbations are considered.

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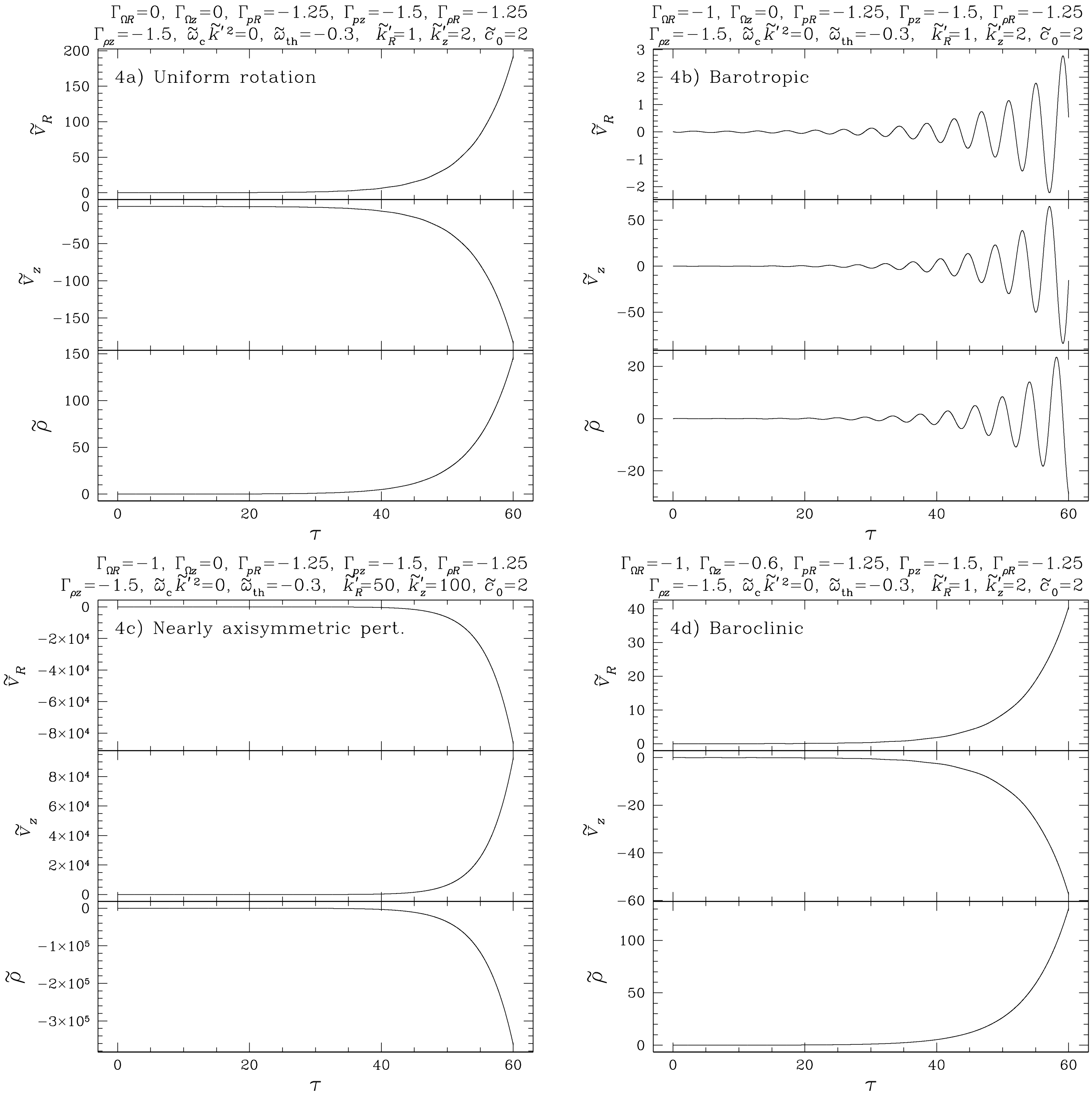

Figure 4. Time evolution of non-axisymmetric velocity and density perturbations for rotating stratified fluids in the presence of radiative cooling ($\tilde{\omega}_{\rm th} = -0.3$), but in the absence of thermal conduction ($\tilde{\omega}_{\rm c} = 0$). In all cases we assume $\tilde{v}_R(0) = 0.01$, $\tilde{v}_z(0) = 0.01$, $\tilde{\rho}(0) = 0.01$, $\tilde{k}'_z/\tilde{k}'_R = 2$, $\tilde{c}_0 = 2$, $\Gamma_{pz} = -1.5$, and $\Gamma_{pR} = -1.25$. In model 4a the system is uniformly rotating ($\Gamma_{\Omega R} = 0, \Gamma_{\Omega z} = 0$), and the perturbation has "low" azimuthal wave-number ($\tilde{k}'_R = 1$). Model 4b is the same as model 4a, but with $\Gamma_{\Omega R} = -1$ (barotropic, differentially rotating, "low" azimuthal wave-number). Model 4c is the same as model 4b, but with $\tilde{k}'_R = 50$ (barotropic, differentially rotating, "high" azimuthal wave-number). Model 4d is the same as model 4b, but with $\Gamma_{\Omega z} = -0.6$ (baroclinic, differentially rotating, "low" azimuthal wave-number). Time is in units of Ω^{-1}.

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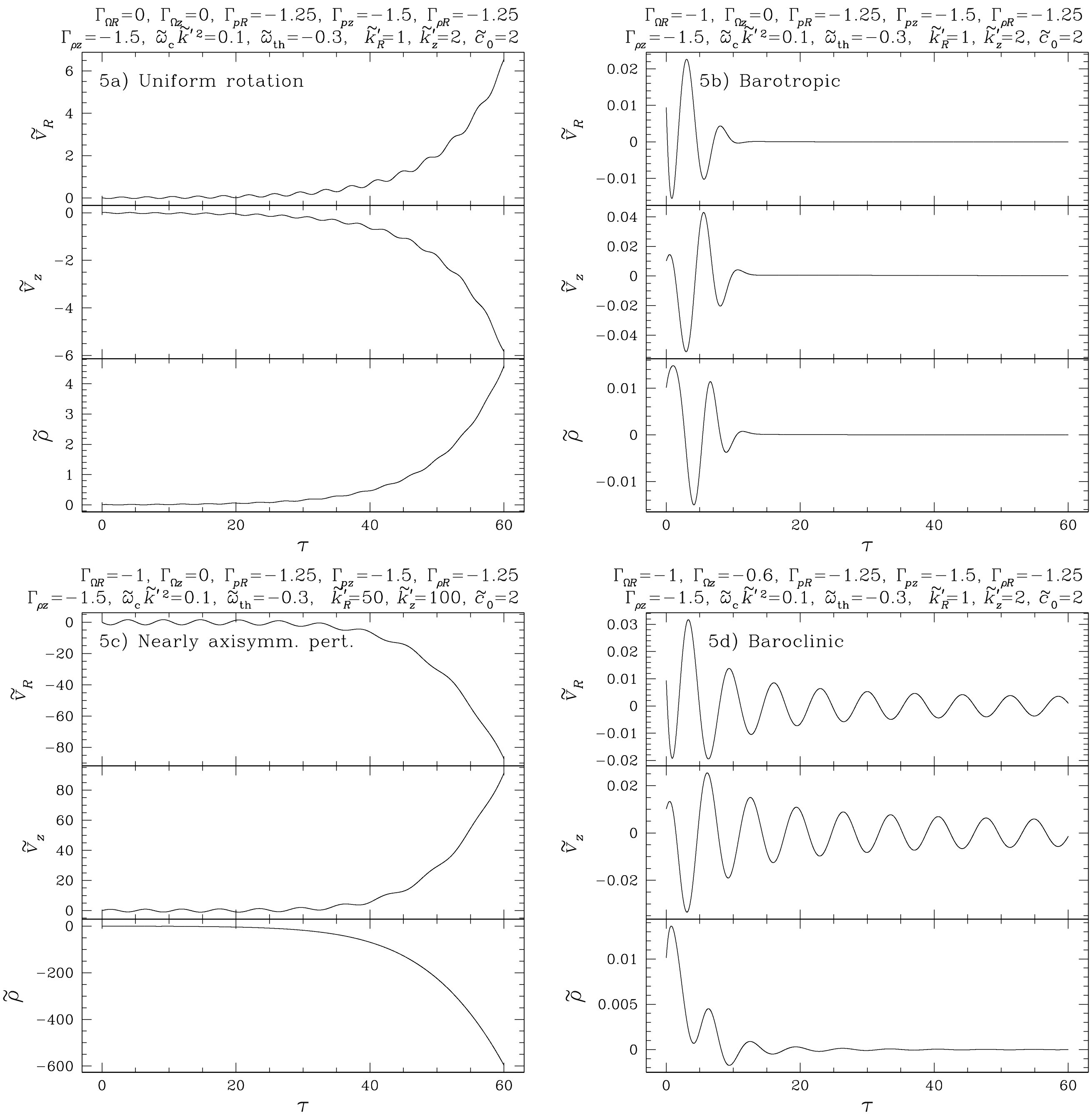

Figure 5. Same as Fig. 4 for models with all the same parameters as the corresponding models in that figure, but with $\tilde{\omega}_c > 0$. In particular, we adopted $\tilde{\omega}_c k'^2 = 0.1$, where $\tilde{k}'^2 \equiv \tilde{k}'^2_R + \tilde{k}'^2_z + 1 = \tilde{k}^2(0)$, so that the initial dissipative frequency $\omega_d(0)$ is negative.

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APPENDIX A: SHORT-WAVELENGTH, LOW-FREQUENCY, AXISYMMETRIC PERTURBATIONS ARE ALMOST ISOBARIC

Here we show that in a differentially rotating fluid short-wavelength, low-frequency, axisymmetric perturbations are almost isobaric, in the sense that $p/p_0 \ll \rho/\rho_0$, where p_0 and ρ_0 are the unperturbed pressure and density, while p and ρ are the pressure and density perturbations. Let us consider the linearized mass and momentum equations (9-12): eliminating v_R, v_z and v_ϕ, and we obtain

$$\frac{p}{p_0} = \frac{\alpha}{\beta}\frac{\rho}{\rho_0}, \tag{A1}$$

where

$$\alpha \equiv \frac{\hat{\omega}^2}{c_0^2 k^2} - \mathrm{i}\frac{\hat{\omega}^2}{\bar{\omega}^2}\frac{k_R A_{pR}}{k^2} - 2\mathrm{i}\frac{\Omega\Omega_z}{\bar{\omega}^2}\frac{k_R A_{pz}}{k^2} - \mathrm{i}\frac{k_z A_{pz}}{k^2} \tag{A2}$$

and

$$\beta \equiv \frac{\hat{\omega}^2}{\bar{\omega}^2}\frac{k_R^2}{k^2} + 2\frac{\Omega\Omega_z}{\bar{\omega}^2}\frac{k_R k_z}{k^2} + \frac{k_z^2}{k^2}, \tag{A3}$$

with $\bar{\omega}^2 = \hat{\omega}^2 - 2\Omega\Omega_R - 2\Omega^2$. In the limit of short wavelengths and low frequencies adopted in Section 3, all terms of α are infinitesimal, while β is finite. It follows $p/p_0 \ll \rho/\rho_0$ and then the perturbed equation of state $\rho/\rho_0 = p/p_0 - T/T_0$ can be approximated as $\rho/\rho_0 \sim -T/T_0$ (see Tribble 1989, for the analogous calculation in the non-rotating case).

APPENDIX B: SOME PROPERTIES OF CUBIC EQUATIONS

We recall here a few properties of cubic equations. Let us consider the equation

$$\tilde{n}^3 + a\tilde{n}^2 + b\tilde{n} + c = 0, \tag{B1}$$

in the unknown $\tilde{n}$, with coefficients a, b and c. The associated discriminant is

$$\Delta = -4a^3c + a^2b^2 - 4b^3 + 18abc - 27c^2. \tag{B2}$$

The roots of equation (B1) have the following properties:

- *Number of real roots.* If $\Delta \geq 0$ the equation has three real roots; if $\Delta < 0$ the equation has one real roots and a pair of complex conjugate roots.
- *Routh-Hurwitz Theorem.* All the roots have negative real parts if and only if the following conditions are satisfied:

$$a > 0, \qquad \begin{vmatrix} a & 1 \\ c & b \end{vmatrix} > 0, \qquad \begin{vmatrix} a & 1 & 0 \\ c & b & a \\ 0 & 0 & c \end{vmatrix} > 0. \tag{B3}$$

- *Viète's formulae.* The three roots $\tilde{n}_1$, $\tilde{n}_2$ and $\tilde{n}_3$ satisfy Viète's formulae:

$$\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 = -a, \tag{B4}$$

$$\tilde{n}_1\tilde{n}_2 + \tilde{n}_2\tilde{n}_3 + \tilde{n}_3\tilde{n}_1 = b, \tag{B5}$$

$$\tilde{n}_1\tilde{n}_2\tilde{n}_3 = -c. \tag{B6}$$